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On the Uniqueness of Planck's Constant

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It is shown that, for the case of two different particles associated with two different Planck constants, an appropriate generalized center-of-mass transformation allows one to retain the relevant constants of the motion. Therefore, some of the problems arising from the (postulated or experimentally determined) existence of several quantization constants appear to be avoidable.

Although it is known that it cannot be logically excluded that Planck's constant $h = 2\pi\hbar$ is not a unique universal one (Wichmann, 1971), not much attention has been paid to this fact owing to the remarkable agreement between theory and experiment reached in quantum physics. In an interesting paper, however, Fischbach *et al.* (1991), after recalling how the existence of several quantization constants leads to an apparent violation of space-time symmetry laws, suggest a possible test of the uniqueness of Planck's constant. It appears therefore worthwhile to explore more closely some of the consequences of the existence of several quantization constants.

In the present paper it shall be shown that the introduction of more than one quantization constant, although in principle possible, is not only unnecessary and avoidable, but also undesirable. If, however, owing to the experimental findings along the lines suggested by Fischbach and co-workers the introduction of multiple Planck constants cannot be avoided, the considerations below show how to avoid some of the problems thereby arising.

To start with, let us recall how Planck's constant is formally introduced in quantum theory. By assuming the validity of the usual space-time symmetries and making use of Wigner's and Stone's theorems (Jordan, 1969), the operators that perform translations in time, translations in space

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(along the x direction, say), and rotations in space (around the z axis, say) are, respectively, the unitary operators $U(t)$, $T_x(\varepsilon)$, and $R_z(\varphi)$ given by

$$U(t) = \exp(-itH/\hbar) \tag{1a}$$

$$T_x(\varepsilon) = \exp(-i\varepsilon P_x/\hbar) \tag{1b}$$

$$R_z(\varphi) = \exp(-i\varphi J_z/\hbar) \tag{1c}$$

In the first of equations (1), the generator H is time independent, for otherwise even the semigroup propriety of the one-parameter set of unitary operators $U(t)$ would not be satisfied. Therefore, when the Schrödinger equation is deduced from the unitary dynamical group the Hamiltonian H is necessarily time independent and the uniformity of time guarantees the conservation of the dynamical variable (to which we agree to give the name of *energy*) associated with the Hermitian operator H. In equations (1b) and (1c), the Hermitian operators P_x and J_z are the generators of the unitary transformations performed by $T_x(\varepsilon)$ and $R_z(\varphi)$, and are *defined* to be the linear momentum x component and the angular momentum z component. If space is homogeneous and isotropic, these quantities are conserved. In order to make the exponents dimensionless, there appears in equations (1) a constant with dimensions of an action, whose numerical value has to be determined from experiments: Planck's constant.

In principle, nothing prevents one from choosing to introduce more than one quantization constant: one for the energy, up to three for the linear momentum, and up to three for the angular momentum. To make things simpler, let us assume that there is only one quantization constant for the three directions of translation and only one for the three rotation axes. Equations (1) are then rewritten as

$$U(t) = \exp(-itH/\hbar') \tag{2a}$$

$$T_x(\varepsilon) = \exp(-i\varepsilon P_x/\hbar'') \tag{2b}$$

$$R_z(\varphi) = \exp(-i\varphi J_z/\hbar''') \tag{2c}$$

Let us see what sorts of consequences would follow from such a choice. First, notice that, since it will appear in the equation of the dynamical evolution, $\hbar'$ can be chosen as the reference Planck's constant, and therefore we shall set $\hbar' \equiv \hbar$. From the fact that rotations in ordinary three-dimensional space do not commute, in general, with each other, it can be shown that, if γ and γ' are infinitesimal rotation angles, then (see, e.g., Battaglia and George, 1990),

$$R_y(-\gamma')\, R_x(\gamma)\, R_y(\gamma')\, R_x(-\gamma) = R_z(\gamma'\gamma) \tag{3}$$

Equations (2c) and (3) give the commutation relations among the angular momentum components (ε_{jkq} is the Levi-Civita symbol)

$$[J_j, J_k] = i\hbar''' \varepsilon_{jkq} J_q \tag{4}$$

which, when applied to the orbital angular momentum, give

$$[X_j, P_k] = i\hbar''' \delta_{jk} \tag{5}$$

Note the appearance in equation (5) of the angular momentum quantization constant. From equations (2b) and (5) one has

$$[X, T_x(\varepsilon)] = T_x(\varepsilon)\varepsilon\hbar'''/\hbar'' \tag{6}$$

and from this and from the eigenvalue equation for the position operator X,

$$X\,|x\rangle = x\,|x\rangle \tag{7}$$

one obtains

$$XT_x(\varepsilon)\,|x\rangle = (x + \varepsilon\hbar'''/\hbar'')\,T_x(\varepsilon)\,|x\rangle \tag{8}$$

In the coordinate representation, and with an appropriate choice of the relative phase among the different eigenvectors of the position operator X, equation (8) is equivalent to

$$T_x(\varepsilon)\,\psi(x) = \psi(x - \varepsilon\hbar'''/\hbar'') \tag{9}$$

We see that a necessary condition for space to be homogeneous is that $\hbar'' = \hbar'''$. In general, the assumption on the validity of the space-time symmetry properties (an assumption that allows us to make use of Wigner's theorem and to derive the time-evolution equation for quantum states) is consistent with (if not implies) the choice of a unique quantization constant [see, however, the interesting new insights provided by Jordan (1992)]. In order to preserve the validity of the homogeneity of space, one could redefine the notions of energy, linear momentum, and angular momentum, as given in equations (2), by means of the replacements

$$P_x \to \hbar'' P_x/\hbar \qquad \text{and} \qquad J_z \to \hbar''' J_z/\hbar \tag{10}$$

thereby obtaining the familiar equations (1).

In the case in which different quantization constants are associated with different particles, one would again lose, and this time unavoidably, the validity of space-time invariance laws. If the particles 1 and 2, whose position and momentum coordinates are $(\mathbf{r}_1, \mathbf{p}_1)$ and $(\mathbf{r}_2, \mathbf{p}_2)$, respectively, interact via a potential V depending only on the separation between 1 and 2, the Hamiltonian is

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + V(\mathbf{r}_1 - \mathbf{r}_2) \tag{11}$$

By performing a center-of-mass transformation,

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{12a}$$

$$\mathbf{p} = \frac{\mu}{m_1}\mathbf{p}_1 - \frac{\mu}{m_2}\mathbf{p}_2 \tag{12b}$$

$$\mathbf{R} = \frac{m_1}{M}\mathbf{r}_1 + \frac{m_2}{M}\mathbf{r}_2 \tag{12c}$$

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 \tag{12d}$$

$$M = m_1 + m_2 \tag{13a}$$

$$\mu = \frac{m_1 m_2}{M} \tag{13b}$$

we find that the above Hamiltonian becomes

$$H = H_R + H_r \tag{14a}$$

with

$$H_R = \frac{P^2}{2M} \qquad \text{and} \qquad H_r = \frac{p^2}{2\mu} + V(\mathbf{r}) \tag{14b}$$

thereby allowing the well-known separation of the equations of the motion. Equations (11)–(14) hold in classical as well as in quantum mechanics, provided the dynamical variables of classical mechanics are promoted to operators obeying a suitable algebra. The quantization prescription requires that we replace the classical Poisson brackets among the canonical conjugate pairs of the position and linear momentum Cartesian coordinates by the commutators (divided by $i\hbar$) among the corresponding linear operators. If one allows for the existence of more than one quantization constant, the canonical commutation relations for the two-particle system considered above are ($j = 1, 2, 3$)

$$[x_{1j}, p_{1j}] = i\hbar_1 \tag{15a}$$

$$[x_{2j}, p_{2j}] = i\hbar_2 \tag{15b}$$

$$\text{any other commutator} = 0 \tag{15c}$$

By combining equations (12) and (15), one obtains the commutation relations among the center-of-mass coordinates. In particular, one obtains

$$[x_j, P_k] = i(\hbar_1 - \hbar_2)\,\delta_{jk} \tag{16}$$

which, together with equations (14), gives

$$[\mathbf{P}, H] = [\mathbf{P}, V(\mathbf{r})] \neq 0 \tag{17}$$

thereby showing that the total linear momentum **P** is not a constant of the motion: Space appears not to be homogeneous.

In the case in which the potential V depends only on the distance between the two particles, the commutation relations between H and the angular momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \tag{18}$$

are

$$[L_q, H] = \varepsilon_{qjk} \frac{p_{1j} p_{2k}}{M}\, i(\hbar_1 - \hbar_2) \neq 0 \tag{19}$$

thereby showing that the angular momentum **L** is not a constant of the motion: Space appears not to be isotropic.

The question therefore arises of whether the problem of the two real interacting particles can be replaced by a problem of two fictitious noninteracting particles to which a unique Planck's constant can be ascribed, the reference Planck's constant which appears in the time-evolution operator. As we shall see below, the answer to the above question turns out to be positive. In what follows it will be shown that, by defining an appropriate center-of-mass transformation (which shall be called a generalized center-of-mass transformation), one can avoid the difficulties mentioned above. Let us, in fact, define the following generalized center-of-mass transformation:

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{20a}$$

$$\mathbf{p} = \frac{\hbar_1 \mu}{\hbar m_1} \mathbf{p}_1 - \frac{\hbar_2 \mu}{\hbar m_2} \mathbf{p}_2 \tag{20b}$$

$$\mathbf{R} = \frac{\hbar_2}{\hbar_1} \left(\frac{m_1 \mu}{m_2 M} \right)^{1/2} \mathbf{r}_1 + \frac{\hbar_1}{\hbar_2} \left(\frac{m_2 \mu}{m_1 M} \right)^{1/2} \mathbf{r}_2 \tag{20c}$$

$$\mathbf{P} = \frac{\hbar_2}{\hbar}\left(\frac{\mu M}{m_1 m_2}\right)^{1/2} \mathbf{p}_1 + \frac{\hbar_1}{\hbar}\left(\frac{\mu M}{m_1 m_2}\right)^{1/2} \mathbf{p}_2 \tag{20d}$$

$$M = m_1 + m_2 \tag{21a}$$

$$\mu = \frac{\hbar^2 m_1 m_2}{(\hbar_2^2 m_1 + \hbar_1^2 m_2)} \tag{21b}$$

It is simple matter to see that the Hamiltonian, in terms of the new set of position and momentum coordinates associated with the fictitious particles of mass M and μ, is still given by equations (14). Moreover, the separation between the center-of-mass and relative coordinates, which, under the algebra specified by equations (15), would not be possible if one performed the transformation defined in equations (12) and (13), is feasible if the generalized center-of-mass transformation were applied, since one would indeed have

$$[H_R, H_r] = 0 \tag{22}$$

In fact, in the presence of two different quantization constants, the usual center-of-mass transformation would give

$$[H_R, H_r] \neq 0 \tag{23}$$

and the two-body problem could not be reduced to a one-body one. Furthermore, the commutation relations among the new coordinates defined by the generalized center-of-mass transformations are

$$[x_j, p_k] = [R_j, P_k] = i\hbar\delta_{jk} \tag{24a}$$

$$\text{any other commutator} = 0 \tag{24b}$$

In particular,

$$[x_j, P_k] = 0 \tag{25}$$

and

$$[\mathbf{P}, H] = [\mathbf{L}, H] = 0 \tag{26}$$

i.e., the quantities $\mathbf{P}$ and $\mathbf{L}$ defined through equations (20) are now conserved.

In conclusion, until an experimental decision is made, one could logically introduce multiple Planck's constants and imagine constructing a physical theory in which different outcomes of two experiments are ascribed to space-time asymmetries rather than to different conditions intrinsic to the system on which the experiments are performed. However, Fischbach and co-workers proposed a valuable approach to deciding experimentally whether Planck's constant is unique or not. If it turns out not to be unique, the usual formulation of quantum mechanics would have to be modified. For instance, as it has been shown here, it is not $\mathbf{p}_1 + \mathbf{p}_2$, given by

$$\mathbf{p}_1 + \mathbf{p}_2 = \frac{\hbar_1 - \hbar_2}{\hbar}\mathbf{p} + \frac{\hbar_2 m_1 + \hbar_1 m_2}{[M(\hbar_2^2 m_1 + \hbar_1^2 m_2)]^{1/2}}\mathbf{P} \tag{27}$$

but **P**, defined in equation (20d), that is the conserved quantity. Remarkably, making use of the same generalized center-of-mass transformation, similar considerations can be made for the angular momentum. The modifications, as we see, might involve a profound revision of concepts and in a manner not yet fully explored (although some suggestions have been proposed by Fischbach and co-workers). However, the difficulties raised for the two-particle case regarding the loss of the usual conserved quantities can be avoided, as has been shown here, by means of the introduction of the generalized center-of-mass transformation. The transformations (20) and (21) are such that (1) with the phase-space coordinates of the system are associated operators obeying the usual canonical commutation relations, and (2) one can maintain the existence of conserved quantities that, even though they do not have classical analogs, can be profitably used in studying the time evolution of the system.

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